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B. Sc. (Honrs) Part 2 Paper 3

Subject Mathematics

Title/Heading of topic :Binary operations

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Binary Operations

Let S be a non-empty set. A function f from $S \times S$ to S is called a binary operation on S i.e. $f : S \times S \rightarrow S$ is a binary operation on set S.

Closure Property

An operation * on a non-empty set S is said to satisfy the closure property, if

$$a \in S, b \in S \Rightarrow a * b \in S, \forall\ a,\ b \in S$$

Also, in this case we say that S is closed for *.

An operation * on a non-empty set S, satisfying the closure property is known as a binary operation.

Properties

(i) Generally binary operations are represented by the symbols * , $\oplus$, ... etc., instead of letters figure etc.

(ii) Addition is a binary operation on each one of the sets N, Z, Q, R and C of natural numbers, integers, rationals, real and complex numbers, respectively. While addition on the set S of all irrationals is not a binary operation.

(iii) Multiplication is a binary operation on each one of the sets N, Z, Q, R and C of natural numbers, integers, rationals, real and complex numbers, respectively. While multiplication on the set S of all irrationals is not a binary operation.

(iv) Subtraction is a binary operation on each one of the sets Z, Q, R and C of integers, rationals, real and complex numbers, respectively. While subtraction on the set of natural numbers is not a binary operation.

(v) Let S be a non-empty set and $P(S)$ be its power set. Then, the union, intersection and difference of sets, on $P(S)$ is a binary operation.

(vi) Division is not a binary operation on any of the sets N, Z, Q, R and C. However, it is not a binary operation on the sets of all non-zero rational (real or complex) numbers.

(vii) Exponential operation $(a, b) \rightarrow a^b$ is a binary operation on set N of natural numbers while it is not a binary operation on set Z of integers.

Types of Binary Operations

(i) **Associative Law** A binary operation * on a non-empty set S is said to be associative, if $(a * b) * c = a * (b * c), \forall\ a, b, c \in S$.

Let R be the set of real numbers, then addition and multiplication on R satisfies the associative law.

(ii) **Commutative Law** A binary operation * on a non-empty set S is said to be commutative, if

$$a * b = b * a, \forall\ a, b \in S.$$

Addition and multiplication are commutative binary operations on Z but subtraction not a commutative binary operation, since

$$2-3 \neq 3-2.$$

Union and intersection are commutative binary operations on the power set $P(S)$ of all subsets of set S. But difference of sets is not a commutative binary operation on $P(S)$.

(iii) **Distributive Law** Let * and o be two binary operations on a non-empty sets. We say that * is distributed over o., if

$a * (b\ o\ c) = (a * b)\ o\ (a * c), \forall\ a, b, c \in S$ also called (left distribution) and $(b\ o\ c) * a = (b * a)\ o\ (c * a), \forall\ a, b, c \in S$ also called (right distribution).

Let R be the set of all real numbers, then multiplication distributes addition on R.

Since, $a \cdot (b+c) = a \cdot b + a \cdot c, \forall\ a, b, c \in R$.

(iv) **Identity Element** Let * be a binary operation on a non-empty set S. An element $e \in S$, if it exist such that

$$a * e = e * a = a, \forall\ a \in S.$$

is called an identity elements of S, with respect to *.

For addition on R, zero is the identity elements in R.

Since, $a + 0 = 0 + a = a, \forall\ a \in R$

For multiplication on R, 1 is the identity element in R.

Since, $a \times 1 = 1 \times a = a, \forall\ a \in R$

Let $P(S)$ be the power set of a non-empty set S. Then, ϕ is the identity element for union on $P(S)$ as

$$A \cup \phi = \phi \cup A = A, \forall\ A \in P(S)$$

Also, S is the identity element for intersection on $P(S)$.

Since, $A \cap S = A \cap S = A, \forall\ A \in P(S)$.

For addition on N the identity element does not exist. But for multiplication on N the identity element is 1.

(v) **Inverse of an Element** Let * be a binary operation on a non-empty set S and let e be the identity element.

Let $a \in S$ we say that a^{-1} is invertible, if there exists an element $b \in S$ such that $a * b = b * a = e$

Also, in this case, b is called the inverse of a and we write, $a^{-1} = b$

Addition on N has no identity element and accordingly N has no invertible element.

Multiplication on N has 1 as the identity element and no element other than 1 is invertible.

Let S be a finite set containing n elements.

Then, the total number of binary operations on S in n^{n^2}.

Let S be a finite set containing n elements.

Then, the total number of commutative binary operation on S is $n^{\frac{n(n+1)}{2}}$.